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Divergence Almost Everywhere of a Pointwise Comparison of Two Sequences of Linear Operators

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This paper deals with a negative result concerning a pointwise comparison of two quite general sequences of linear operators on the space of continuous functions on the torus or a segment

Let E be the torus or a nongenerate segment of $\mathbb{R}$. μ denotes the Lebesque measure on E and $\|\cdot\|$ denotes the uniform norm on the space of continuous functions on E. Shapiro [6, p. 120] set up a problem of comparative pointwise behaviour of two sequences of linear operators on $C(E)$. This question was studied in many works. For different pairs $\{A_n\}$ and $\{B_n\}$ of sequences of operators it was shown that the relationship

$$|B_n(f)-f| = O(|A_n(f)-f|)$$

may fail almost everywhere (see [1–4]). Below we prove that this occurs for quite general sequences of operators.

THEOREM. *Let $\{A_n\}$ and $\{B_n\}$ be sequences of finite-dimensional linear bounded operators from the space $C(E)$ to itself and let $\{I_n\}$ be a non-decreasing sequence of positive numbers. Suppose also that $A_n(f) \to f$ uniformly for any f belonging to a dense subset of $C(E)$ and for some sequence of functions $h_n \in C(E)$ we have*

$$\lim_{n\to\infty}\inf \mu\{x \in E\colon A_n(h_n; x) = B_n(h_n; x)\} = 0.$$

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Then there exists a function $f \in C(E)$ and a sequence $\{n_k\}$ such that

$$\lim_{k \to \infty} I_{n_k} |A_{n_k}(f) - f| / |B_{n_k}(f) - f| = 0$$

almost everywhere.

The theorem contains some previous results. For instance, if one of the operators A_n, B_n is the trigonometric convolution operator and another operator is its discrete analogue, $I_n = 1$; then the assertion follows from [2, 3]. However, the analogous result, obtained in [1] for the Abel–Poussin means A_n, the Fejér means B_n, and $I_n = o(\sqrt{\ln n})$, is not covered by our theorem, because A_n are not finite-dimensional operators.

It might also be of interest to state sufficient conditions for the stronger conditions that there exists a continuous function f such that

$$\lim_{n \to \infty} I_n |A_n(f) - f| / |B_n(f) - f| = 0$$

almost everywhere.

To prove the theorem we need two lemmas.

LEMMA 1. *Let $\varepsilon > 0$, L be a finite-dimensional linear space, and let C be a linear operator from $C(E)$ to L. Then there exists a function $h \in C(E)$ such that $Ch = 0$, $\|h\| \leqslant 1/\varepsilon$, and $\mu\{x \in E: h(x) \neq 1\} < \varepsilon \mu E$.*

Proof. The assertion is trivial for $\varepsilon > 1$ since one can take $h = 0$. Let $\varepsilon \leqslant 1$. Denote $n = [1/\varepsilon]$ and $m = \dim L$; then $\varepsilon > 1/(n+1)$ and we can choose $\delta > 0$ and positive integer M such that

$$2\delta < \varepsilon - \frac{1}{n+1} \tag{1}$$

and

$$\frac{m+1}{M} < \delta. \tag{2}$$

We divide E into M equal segments Δ_i. Let h_i $(i = 1, ..., M)$ be a continuous function supported on Δ_i such that

$$\|h_i\| = 1 \tag{3}$$

and

$$\mu\{x \in \Delta_i: h_i(x) \neq 1\} \leqslant \delta \mu \Delta_i. \tag{4}$$

We consider the cube $K = [-n, 1]^M$ in the M-dimensional space $Y = \mathbb{R}^M$, Denote

$$C'(u) = \left(\sum_{i=1}^{M} u_i,\, C\left(\sum_{i=1}^{M} u_i h_i \right) \right) \in \mathbb{R} \oplus L$$

$$(u = (u_1, ..., u_M) \in Y), \quad Z = \operatorname{Ker} C'.$$

Then Z is the subspace of Y and

$$\operatorname{codim} Z = \dim \Im C' \leqslant 1 + \dim L = m + 1.$$

By the Krein–Milman theorem [5, Chap. 1, Sect. 4] $K \cap Z$ contains an extremal point $u = (u_1, ..., u_M)$. We assert that all coordinates of u but at most $m+1$ exceptions are equal 1 or $-n$. Actually, in the opposite case u is an interior point of some face F of K, $\dim F > m+1$. Then $F \cap Z$ contains a segment of a line passing through the point u and so u is not an extremal point that does not agree with our supposition. Since $\sum_{i=1}^{M} u_i = 0$ the number of coordinates $u_i = -n$ does not exceed $M/(n+1)$. Therefore,

$$^{\#}\{i: u_i \neq 1\} = {}^{\#}\{i: u_i = -n\} + {}^{\#}\{i: -n < u_i < 1\} \leqslant \frac{M}{n+1} + m + 1. \qquad (5)$$

Denote $h = \sum_{i=1}^{M} u_i h_i$ and verify that h satisfies the requirements of the lemma. By definition of Z we have $Ch = 0$. Further, since the supports of functions h_i are mutually nonoverlapping segments, $\|h\| = \max_i \|u_i h_i\| = \max_i |u_i|$ by (3). Since $u \in K$ we get $\max_i |u_i| \leqslant n \leqslant 1/\varepsilon$. Finally,

$$\mu\{x \in E: h(x) \neq 1\} \leqslant \sum_{u_i \neq 1} \mu\Delta_i + \sum_{i=1}^{M} \mu\{x \in \Delta_i: h_i(x) \neq 1\}$$

and using consequently (5), (4), (2), and (1) we have

$$\begin{aligned} \mu\{x \in E: h(x) \neq 1\} &\leqslant \left(\frac{M}{n+1} + m + 1 \right) \frac{\mu E}{M} + \delta \sum_{i=1}^{M} \mu\Delta_i \\ &= \left(\frac{1}{n+1} + \frac{m+1}{M} \right) \mu E + \delta\mu E \\ &\leqslant \left(\frac{1}{n+1} + 2\delta \right) \mu E < \varepsilon\mu E \end{aligned}$$

as required. Lemma 1 is proved.

LEMMA 2. *Under the assumptions of the theorem for any* $f \in C(E)$, $N \in \mathbb{N}$, *and* $\varepsilon > 0$ *there exist* $g \in C(E)$, $n > N$, *and* $F \subset E$ *such that* $\|f - g\| \leqslant \varepsilon$, $\mu F \geqslant (1-\varepsilon)\,\mu E$, $A_n g(x) = g(x)$ *for* $x \in F$, *and* $\inf_{x \in F} |g(x) - B_n g(x)| > 0$.

Proof. We consider that $\varepsilon < 1$. By the assumptions of the theorem we can find $f_0 \in C(E)$ and $n > N$ satisfying the conditions

$$\|f - f_0\| \leqslant \varepsilon^2/8, \tag{6}$$

$$\|A_n(f_0) - f_0\| \leqslant \varepsilon^2/8, \tag{7}$$

and

$$\exists h_n \in C(E)\colon \mu(G) < \varepsilon \mu E/2, \tag{8}$$

where $G = \{x \in E\colon A_n(h_n; x) = B_n(h_n; x)\}$. For almost all λ we have

$$\mu\{x \in E \backslash G\colon A_n(f_0 + \lambda h_n; x) = B_n(f_0 + \lambda h_n; x)\} = 0. \tag{9}$$

Therefore we can choose λ satisfying (9) small enough so that $\|\lambda h_n\| \leqslant \varepsilon^2/8$ and $\|A_n(\lambda h_n)\| \leqslant \varepsilon^2/8$. Denote $f_1 = f_0 + \lambda h_n$ and $f_2 = A_n(f_1) - f_1$. We have

$$\|f - f_1\| \leqslant \|f - f_0\| + \|f_0 - f_1\| \leqslant \varepsilon^2/4, \tag{10}$$

$$\|f_2\| \leqslant \|A_n(f_1) - A_n(f_0)\| + \|A_n(f_0) - f_0\| + \|f_0 - f_1\| \leqslant 3\varepsilon^2/8 \tag{11}$$

by (6) and (7) and

$$\mu\{x \in E\colon A_n(f_1; x) = B_n(f_1; x)\} < \varepsilon \mu E/2 \tag{12}$$

by (8) and (9). Let us define the finite-dimensional linear $L = \operatorname{Im} A_n \oplus \operatorname{Im} B_n$ and the linear operator $C\colon C(E) \to L\colon Ch = A_n(f_2 h) \oplus B_n(f_2 h)$. Applying Lemma 1 (with $\varepsilon/2$ instead of ε) we obtain the existence of a corresponding function h. Let $g = f_1 + f_2 h$ and $F' = \{x \in E\colon h(x) = 1\}$; then we have:

(1) $\|f - g\| \leqslant \|f - f_1\| + \|f_1 - g\| \leqslant \|f - f_1\| + 2\,\|f_2\|/\varepsilon$ and by (10) and (11) $\|f - g\| \leqslant \varepsilon^2/4 + 3\varepsilon/4 < \varepsilon$;

(2) $A_n g(x) = A_n f_1(x) = f_1(x) + f_2(x) = g(x)$ for $h(x) = 1$, i.e., $x \in F'$;

(3) $\mu F' > (1 - \varepsilon/2)\,\mu E$;

(4) $A_n g = A_n f_1$, $B_n g = B_n f_1$, and by (12) $\mu\{x \in E\colon A_n(g; x) = B_n(g; x)\} < \varepsilon/2 \mu E$.

Thus, we get

$$\mu\{x \in E: g(x) = A_n(g; x) \neq B_n(g; x)\}$$
$$\geqslant \mu F' - \mu\{x \in E: A_n(g; x) = B_n(g; x)\} > (1-\varepsilon)\,\mu E,$$

and a required set F can be chosen. This completes the proof of Lemma 2.

Now we are ready to prove the theorem. Using Lemma 2 one can easily construct by induction sequences of positive numbers ε_k, positive integers n_k, sets $F_k \subset E$, and functions $f_k \in C(E)$ satisfying for $k = 1, 2, ...,$ the following conditions:

$$\mu F_k \geqslant (1-\varepsilon_k)\,\mu E, \tag{13}$$

$$A_{n_k} f_k(x) = f_k(x) \qquad \text{for} \quad x \in F_k, \tag{14}$$

$$\delta_k = \inf_{x \in F_k} |f_k(x) - B_{n_k} f_k(x)| > 0, \tag{15}$$

$$\|f_k - f_{k+1}\| \leqslant \varepsilon_{k+1}, \tag{16}$$

$$n_{k+1} > n_k, \tag{17}$$

$$\varepsilon_{k+1} < \varepsilon_k/2, \tag{18}$$

$$2\varepsilon_{k+1} < \frac{\delta_k}{kI_{n_k} + 1 + kI_{n_k}\|A_{n_k}\| + \|B_{n_k}\|}. \tag{19}$$

It follows from (18) that

$$\sum_{k=1}^{\infty} \varepsilon_k < \infty. \tag{20}$$

Therefore, by (16), the sequence $\{f_k\}$ uniformly converges to a function $f \in C(E)$ and also

$$\|f - f_k\| < 2\varepsilon_{k+1}. \tag{21}$$

Denote

$$F = \bigcup_{l=1}^{\infty} \bigcap_{k=l}^{\infty} F_k;$$

then from (20) we get $\mu(E \backslash F) = 0$. For almost all points $x \in E$ we have $x \in F$. Then $x \in F_k$ for sufficiently large k and we can use (21), (14), and (15) for estimates of deviations $A_{n_k} f(x) - f(x)$ and $B_{n_k} f(x) - f(x)$. We find

$$|A_{n_k} f(x) - f(x)| \leqslant 2(1 + \|A_{n_k}\|)\,\varepsilon_{k+1}$$

and

$$|B_{n_k} f(x) - f(x)| \geqslant \delta_k - 2(1 + \|B_{n_k}\|)\, \varepsilon_{k+1}.$$

From the last two inequalities and (19) we obtain

$$kI_{n_k} |A_{n_k} f(x) - f(x)| \leqslant |B_{n_k} f(x) - f(x)|.$$

Thus,

$$\lim_{k \to \infty} I_{n_k} |A_{n_k} f(x) - f(x)|/|B_{n_k} f(x) - f(x)| = 0$$

for $x \in F$, and the theorem is proved.

REFERENCES

1. W. Dickmeis, R. J. Nessel, and E. van Wickeren, The divergence almost everywhere of a pointwise comparison of Fejér and Abel–Poisson means, *Bull. London Math. Soc.* **18** (1986), 606–608.
2. R. Getsadze, On one problem of N. Kirchoff and R. J. Nessel, *J. Approx. Theory* **80** (1995), 409–422.
3. N. Kirchoff and R. J. Nessel, Divergence almost everywhere of a pointwise comparison of trigonometric convolution processes with their discrete analogues, *J. Approx. Theory* **70** (1992), 29–38.
4. N. Kirchoff and R. J. Nessel, Convolution processes of Fejér type and the divergence almost everywhere of a pointwise comparison, *Acta Math. Hungar.* **62** (1993), 97–110.
5. V. K. Leichtweiss, "Konvexe Mengen," VEB Deutscher Verlag der Wissenschaften, Berlin, 1980.
6. H. S. Shapiro, "Smoothing and Approximation of Functions," Van Nostrand, New York, 1969.